EINDHOVEN UNIVERSITY OF TECHNOLOGY

Network identifiability - Analysis

Paul Van den Hof

Doctoral School Lyon, France, 11-12 April 2024

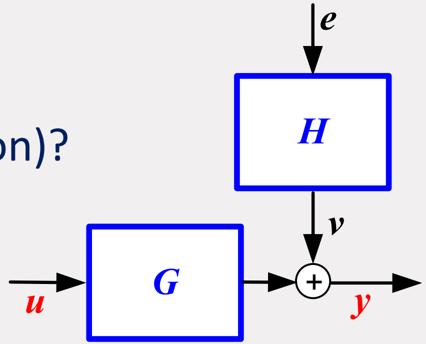
www.sysdynet.eu
www.pvandenhof.nl
p.m.j.vandenhof@tue.nl

European Research Council

Network identifiability

- Introduction – background starting from the open-loop case
- Definition(s) of network identifiability
- Two technical results / conditions for evaluating identifiabiltiy
- Generic identifiability through path-based graph conditions
- Discussion and Summary

Introduction – classical situation



When are models essentially different (in view of identification)?

In classical PE identification:
Models are indistinguishable (from data) if their predictor filters are the same:

$$\hat{y}(t|t-1) = \underbrace{H(q)^{-1}G(q)}_{W_u(q)}\, u(t) + \underbrace{[1 - H(q)^{-1}]}_{W_y(q)}\, y(t)$$

Introduction – classical situation

predictor: $\hat{y}(t|t-1;\theta_1) = \hat{y}(t|t-1;\theta_2)$

Two models can not be distinguished from data

↓ → data informativity for model set (Ljung, 1999)

predictor filters:
$$\begin{aligned} W_y(q,\theta_1) &= W_y(q,\theta_2) \\ W_u(q,\theta_1) &= W_u(q,\theta_2) \end{aligned}$$

↓ → trivial for classical (open-loop) case

models:
$$\begin{aligned} G(q,\theta_1) &= G(q,\theta_2) \\ H(q,\theta_1) &= H(q,\theta_2) \end{aligned}$$

↓ → Identifiability of model structure

parameters: $\theta_1 = \theta_2$

Introduction – classical situation

predictor: $\hat{y}(t|t-1;\theta_1) = \hat{y}(t|t-1;\theta_2)$

↓

predictor filters:
$$\begin{aligned} W_y(q,\theta_1) &= W_y(q,\theta_2) \\ W_u(q,\theta_1) &= W_u(q,\theta_2) \end{aligned}$$

↓ → **Non-trivial for network case**

models:
$$\begin{aligned} G(q,\theta_1) &= G(q,\theta_2) \\ H(q,\theta_1) &= H(q,\theta_2) \end{aligned}$$

↓

parameters: $\theta_1 = \theta_2$

Reason:

- Freedom in network structure
- Freedom in presence of excitations and disturbances

TU/e

Network identifiability problem

The network **model**:

$$w(t) = G(q)w(t) + R(q)r(t) + H(q)e(t)$$

can be transformed with any rational $P(q)$:

$$P(q)w(t) = P(q)\{G(q)w(t) + R(q)r(t) + G(q)e(t)\}$$

to an **equivalent model**:

$$w(t) = \tilde{G}(q)w(t) + \tilde{R}(q)r(t) + \tilde{H}(q)\tilde{e}(t)$$

$\Longrightarrow$ **Nonuniqueness**, unless there are structural constraints on G, R, H.

TU/e

Network identifiability problem

How can we formalize this problem?

Based on measured data w and r, and knowing that r will always be an input, i.e. there is no effect from w on r,
we can write the network expression: $w(t) = T_{wr}(q)r(t) + \underbrace{T_{we}(q)e(t)}_{\bar{v}(t)}$

with $T_{wr} = (I-G)^{-1}R$ and $T_{we} = (I-G)^{-1}H$.

This implies that the information that maximally can be extracted[1] from w, r can be represented by $(T_{wr}, \Phi_{\bar{v}})$

So the network identifiability question can be phrased as:

Is there a unique map from $(T_{wr}, \Phi_{\bar{v}})$ to (G, R, H)?

[1] Based on second order signal properties

Network identifiability problem

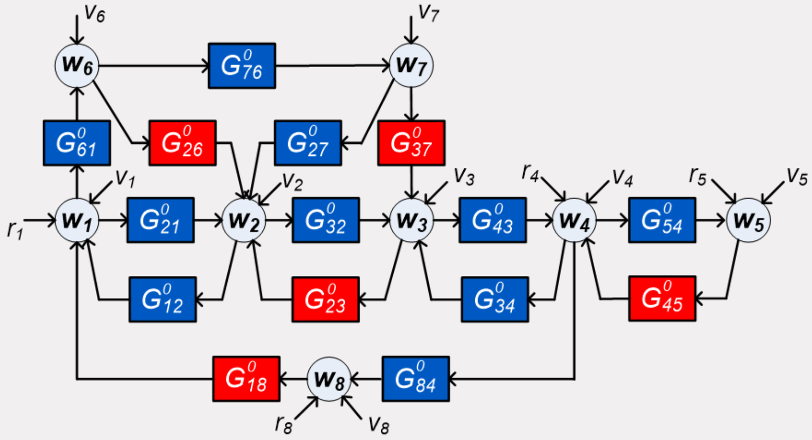
$$w(t) = T_{wr}(q)r(t) + \underbrace{T_{we}(q)e(t)}_{\bar{v}(t)}$$

Is there a unique map from $(T_{wr}, \Phi_{\bar{v}})$ to (G, R, H)?

What would happen if this mapping is non-unique?

Then there are different network models (G_1, R_1, H_1), (G_2, R_2, H_2)
with possibly different network topologies,
that generate the same $(T_{wr}, \Phi_{\bar{v}})$,
and thus cannot be distinguished from measured (w, r).

Network identifiability



blue = unknown/parametrized
red = fixed/known

- Like in “classical” identification we apply the identifiability concept to a model set
- Network identifiability can be achieved by having a sufficient number of restrictions in the model set, e.g. on the topology of G
- In the parametrized model set some elements can be fixed (because they are assumed to be known a priori)

Network identifiability

Network: $w = G^0 w + R^0 r + H^0 e$ $\quad cov(e) = \Lambda^0,$ rank *p*

dim(*r*) = *K*

The network is defined by: $(G^0, R^0, H^0, \Lambda^0)$

a network model is denoted by: $M = (G, R, H, \Lambda)$

and a **network model set** by:

$$\mathcal{M} = \{M(\theta) = (G(\theta), R(\theta), H(\theta), \Lambda(\theta)), \theta \in \Theta\}$$

represents **prior knowledge** on the network models:

- topology
- disturbance correlation
- known (non-parametrized) modules
- external excitation signals available

TU/e

Network identifiability

Definition Network identifiability[1]

For a network model set $\mathcal{M}$, consider a model $M(\theta_0) \in \mathcal{M}$ and the implication

$$\left.\begin{array}{c} T_{wr}(q,\theta_0) = T_{wr}(q,\theta_1) \\ \Phi_{\bar{v}}(\omega,\theta_0) = \Phi_{\bar{v}}(\omega,\theta_1) \end{array}\right\} \Longrightarrow \{\ M(\theta_0) = M(\theta_1),$$

for all $M(\theta_1) \in \mathcal{M}$

Then $\mathcal{M}$ is

- globally identifiable from (w,r) at $M(\theta_0)$ if the implication holds for $M(\theta_0)$;
- globally identifiable from (w,r) if it holds for all $M(\theta_0) \in \mathcal{M}$;
- generically identifiable[2] from (w,r) if it holds for almost all $M(\theta_0) \in \mathcal{M}$;

[1] Weerts et al., Automatica, March 2018;
[2] Hendrickx et al., IEEE-TAC, 2019.

[3] Legat and Hendrickx, CDC 2020, present a local version, where the implication above is requested to hold only in a neighborhood of $M(\theta_0)$

TU/e

Network identifiability

The pair of objects $(T_{wr}, \Phi_{\bar{v}})$ plays a central role

It would be attractive (for analysis) to consider the pair (T_{wr}, T_{we})

Under which conditions does $\Phi_{\bar{v}} = (I-G)^{-1}H\Lambda H^*(I-G)^{-*}$ provide a unique $T_{we} = (I-G)^{-1}H$?

e.g. if $(I-G)^{-1}H$ is monic then spectral factorization of $\Phi_{\bar{v}}$ provides a unique T_{we}

Proposition

If
1. The modules in $G(\theta)$ are strictly proper, or
2. No algebraic loops in $G(\theta)$ and $H^\infty(\theta)\Lambda(\theta)H^\infty(\theta)^T$ is diagonal, with $H^\infty(\theta) := \lim_{z\to\infty} H(z,\theta)$

Then $\{T_{wr}, \Phi_{\bar{v}}\} \Leftrightarrow \{T_{wr}, T_{we}, \Lambda\}$

[1] Weerts et al., Automatica, March 2018;

Network identifiability

Explanation

No algebraic loops in $G(\theta) \Longrightarrow$

By row and column permutations, $G^\infty(\theta)$ can be turned into an upper triangular matrix

Then $(I - G^\infty)^{-1}$ has ones on the diagonal $\Longrightarrow$

With $\Phi_{\bar{v}}^\infty = (I - G^\infty)^{-1} \underbrace{H^\infty \Lambda (H^\infty)^T}_{\text{diagonal}} (I - G^\infty)^{-T}$ and H monic,

This fixes Λ when given $\Phi_{\bar{v}}^\infty$
and removes all scaling freedom in the spectral factorization on $\Phi_{\bar{v}}$ ■

Network identifiability

If the conditions of the proposition are satisfied, then the implication in the identifiability definition can be turned into:

$$\left.\begin{aligned} T_{wr}(q,\theta_1) &= T_{wr}(q,\theta_0) \\ T_{we}(q,\theta_1) &= T_{we}(q,\theta_0) \\ \Lambda(\theta_1) &= \Lambda(\theta_0) \end{aligned}\right\} \Longrightarrow M(\theta_1) = M(\theta_0)$$

or equivalently:

$$\left.\begin{aligned} T_{wr}(q,\theta_1) &= T_{wr}(q,\theta_0) \\ T_{we}(q,\theta_1) &= T_{we}(q,\theta_0) \end{aligned}\right\} \Longrightarrow (G(\theta_1), R(\theta_1), H(\theta_1)) = (G(\theta_0), R(\theta_0), H(\theta_0))$$

Network identifiability

Network identifiability of $\mathcal{M}$ from (w,r) is determined by the implication

$$\left.\begin{array}{l} T_{wr}(q,\theta_1) = T_{wr}(q,\theta_0) \\ T_{we}(q,\theta_1) = T_{we}(q,\theta_0) \end{array}\right\} \Longrightarrow (G(\theta_1), R(\theta_1), H(\theta_1)) = (G(\theta_0), R(\theta_0), H(\theta_0))$$

for all $M(\theta_1) \in \mathcal{M}$

- Network identifiability is a property of a parametrized model set
- It is not dependent on any identification method
- It focusses on uniquenes of network models, rather than of parameters

Network identifiability

Different results for network identifiability

- (Conservative) result that is independent of the structure in $G(\theta)$
- More technical result that builds on the structure in $G(\theta)$
- Path-based result on the network graph for generic identifiability

First (conservative) network identifiability result

Denote $U(\theta) := \begin{bmatrix} R(\theta) & H(\theta) \end{bmatrix}$

and $T_{wu} = \begin{bmatrix} T_{wr} & T_{we} \end{bmatrix}$

Then $$T_{wu} = (I - G(\theta))^{-1} U(\theta)$$

and $$(I - G(\theta)) T_{wu} = U(\theta)$$

Prime identifiability question:

Do $G(\theta), U(\theta)$ uniquely follow from T_{wu}?

$U(q, \theta) \in \mathbb{R}(q)^{L \times (K+p)}$ where $K + p$ is the number of external $r + e$ signals.

First (conservative) network identifiability result

Sufficient condition for network identifiability[1],[2] – full excitation case

Consider model set $\mathcal{M}$, and let $U(q,\theta)$ be full row rank $\forall\theta$.
Then $\mathcal{M}$ is globally network identifiable from (r,w) if there exists a nonsingular and parameter-independent matrix $Q(q) \in \mathbb{R}^{(K+p)\times(K+p)}$ such that

$$U(q,\theta)Q(q) = \begin{bmatrix} D(q,\theta) & F(q,\theta) \end{bmatrix}$$

with $D(q,\theta)$ diagonal and full rank for all θ.

- Rank condition on $U(q,\theta)$ implies that $K+p \geq L$, i.e. there are at least as many external signals as there are nodes (full excitation)
- The resulting condition is independent of the structure in $G(q,\theta)$.

[1] Goncalves and Warnick, 2008;
[2] Weerts et al, Automatica, March 2018.;

Network identifiability

Reasoning

$$(I - G(\theta))T_{wu}Q = U(\theta)Q$$

$$(I - G(\theta))T_{wu}Q = \begin{bmatrix} D(\theta) & F(\theta) \end{bmatrix}$$

With $T_{wu}Q = \begin{bmatrix} A & B \end{bmatrix}$ and A full rank, it follows that

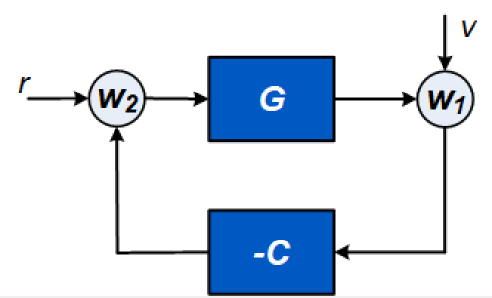
$$\begin{aligned} D(\theta)^{-1}(I - G(\theta))A &= I \\ (I - G(\theta))B &= F(\theta) \end{aligned}$$

Since $D(\theta)$ is diagonal and $I - G(\theta)$ is hollow, uniqueness of $D(\theta)$ and $G(\theta)$ follows.
Then also $F(\theta)$ is unique.

■

TU/e

Example 1



Parametrized model set $\mathcal{M}$ with

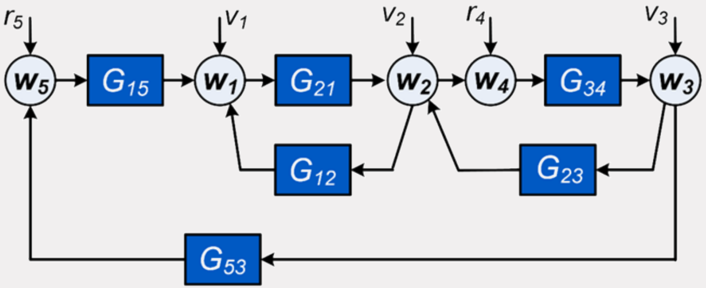
$$G(\theta) = \begin{bmatrix} 0 & G(\theta) \\ -C(\theta) & 0 \end{bmatrix}, \quad R(\theta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H(\theta) = \begin{bmatrix} H(\theta) \\ 0 \end{bmatrix}$$

$U(\theta) = \begin{bmatrix} 0 & H(\theta) \\ 1 & 0 \end{bmatrix}$ can be made diagonal by elementary column operations

$\Longrightarrow$ $\mathcal{M}$ is globally network identifiable. ■

TU/e

Example 2



Consider a model set $\mathcal{M}$ where v_1 and v_2 are allowed to be correlated:

$$\mathcal{M} \text{ with } H(\theta) = \begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) & 0 \\ H_{21}(\theta) & H_{22}(\theta) & 0 \\ 0 & 0 & H_{33}(\theta) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

There is enough excitation, but U can not be transformed to a diagonal matrix.

$\Longrightarrow$ No conclusion that holds for any choice of $G(\theta)$

TU/e

Interpretation

Interpretation of result:

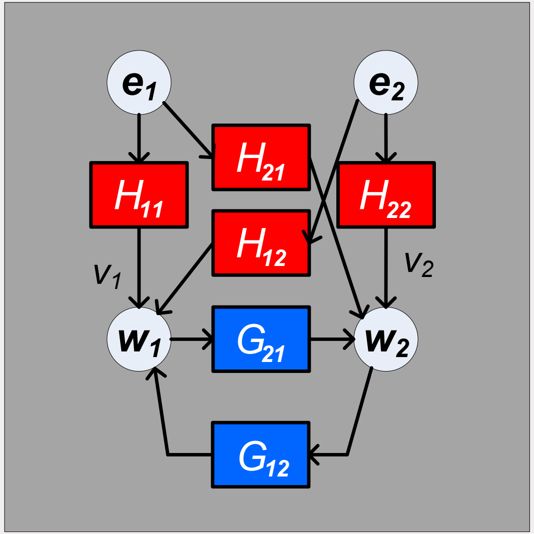
Diagonalizability of $U(\theta)$ is implied by:
having independent external signals at every node

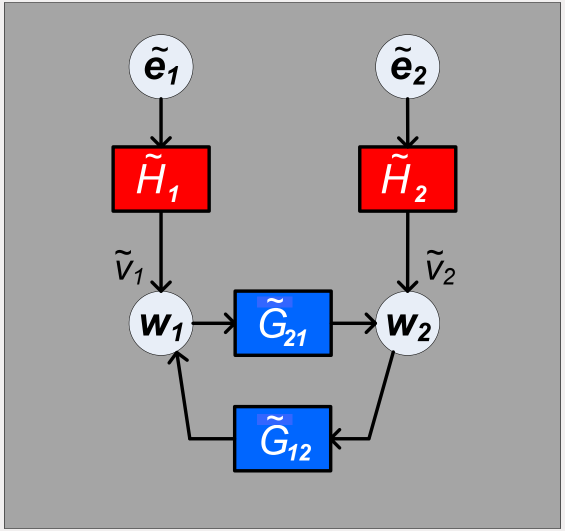
Consequence:

Given data from any LTI dynamic network, there always exists a representing model with diagonal H

But this does not necessarily represent the structured network that has generated the data

Dynamic network setup - nonuniqueness





Node signals $w_1(t), w_2(t)$ being invariant

[1] G. Bottegal et al., SYSID2018

Second network identifiability result

Towards a more general result that takes account of the structure of $G(\theta)$:

$$(I - G(\theta))T_{wu} = U(\theta)$$

Do $G(\theta), U(\theta)$ uniquely follow from T_{wu}?

Consider row j of this equation.
Reorder the columns of $(I - G(\theta))$ and $U(\theta)$ such that

$$\begin{bmatrix} G_1(\theta) & G_2 \end{bmatrix}_{j\star} PT_{wu} = \begin{bmatrix} U_1 & U_2(\theta) \end{bmatrix}_{j\star} Q \qquad P, Q \text{ permutation matrices}$$

Then

$$[G_1(\theta) \quad G_2]_{j\star} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [U_1 \quad U_2(\theta)]_{j\star} \quad \text{with} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} = PT_{wu}Q^{-1}$$

$\Longrightarrow$ $G_1(\theta)_{j\star}, U_2(\theta)_{j\star}$ are uniquely determined if A has full row rank.

TU/e

Second network identifiability result

Sufficient condition for network identifiability[1] – general case

Consider model set $\mathcal{M}$, and define for each $j \in [1, L]$:

$\breve{T}_j :=$ the transfer function from

- all external signals (r, e) that do not enter w_j through a parametrized module, to
- all node signals w that map to w_j through a parametrized module.

Then $\mathcal{M}$ is globally network identifiable from (r, w) if for all $j \in [1, L]$:

$\breve{T}_j$ is full row rank for all $\theta \in \Theta$.

The result allows for $K + p < L$ and distinguishes between parametrized and non-parametrized (fixed) modules in $\mathcal{M}$.

[1] Weerts et al, Automatica, March 2018.

Second network identifiability result

An immediate consequence of the condition is that

$$\sharp \text{ parametrized entries in } \begin{bmatrix} G(\theta) & R(\theta) & H(\theta) \end{bmatrix}_{j\star} \leq K + p$$

Proof:

Follows directly from full row rank condition on $\breve{T}_j$:

$$\sharp \text{ param } G(\theta)_{j\star} \leq K + p - \sharp \text{ param } [R(\theta)\ H(\theta)]_{j\star} \qquad \blacksquare$$

[1] Weerts et al, Automatica, March 2018.

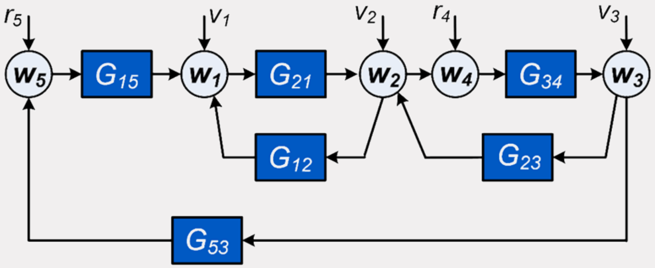
Second network identifiability result – using *G*-structure

The condition becomes also necessary if we add some conditions on $\mathcal{M}$:

- All parametrized entries in $\mathcal{M}$ are parametrized independently, and
- Each parametrized entry in $\mathcal{M}$ is not limited in order, and
- Regularity condition on the fixed/non-parametrized modules

Weerts et al., Automatica, March 2018; Shi et al. Automatica, March 2022.

Example 5-node network (continued)



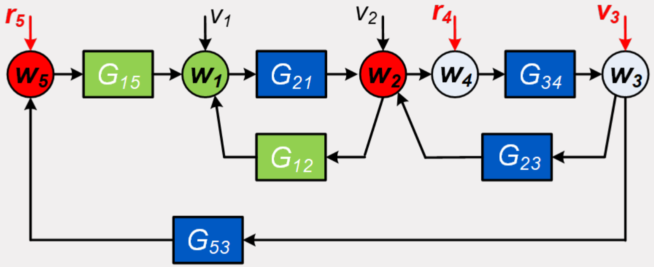
If we restrict the structure of $G(\theta)$:

$$G(\theta) = \begin{bmatrix} 0 & G_{12}(\theta) & 0 & 0 & G_{15}(\theta) \\ G_{21}(\theta) & 0 & G_{23}(\theta) & 0 & 0 \\ 0 & 0 & 0 & G_{34}(\theta) & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & G_{53}(\theta) & 0 & 0 \end{bmatrix} \qquad [H\ \ R] = \underbrace{\begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) & 0 & 0 & 0 \\ H_{21}(\theta) & H_{22}(\theta) & 0 & 0 & 0 \\ 0 & 0 & H_3(\theta) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{K+p=5}$$

First check:

Number of parametrized entries in each row $< K + p = 5$ ✓

Example 5-node network (continued)



Rank condition:

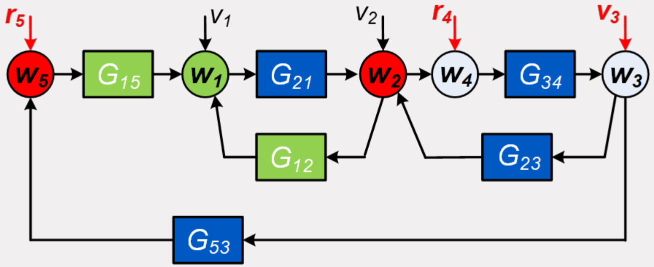
evaluation of $\breve{T}_j$ for $j = 1$:

$$G(\theta) = \begin{bmatrix} 0 & G_{12}(\theta) & 0 & 0 & G_{15}(\theta) \\ G_{21}(\theta) & 0 & G_{23}(\theta) & 0 & 0 \\ 0 & 0 & 0 & G_{34}(\theta) & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & G_{53}(\theta) & 0 & 0 \end{bmatrix}$$

$$[H \;\; R] = \underbrace{\begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) & 0 & 0 & 0 \\ H_{21}(\theta) & H_{22}(\theta) & 0 & 0 & 0 \\ 0 & 0 & H_3(\theta) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{K+p=5}$$

$\breve{T}_1 : \begin{bmatrix} v_3 \\ r_4 \\ r_5 \end{bmatrix} \rightarrow \begin{bmatrix} w_2 \\ w_5 \end{bmatrix}$ has to have full row rank $\forall \theta \in \Theta$

Example 5-node network (continued)



Issues:

- Such a rank test is not easy to apply
- and needs to be done for every $j = 1, \cdots L$

Alternative:

- Evaluate the rank tests for the "generic" case, i.e.
  independent of the particular numerical values of the several transfer functions

Generic identifiability

Generic rank and vertex disjoint paths[1],[2],[3]

The **generic rank** of a transfer function matrix between

inputs $\boldsymbol{u}$ and nodes $\boldsymbol{w}$

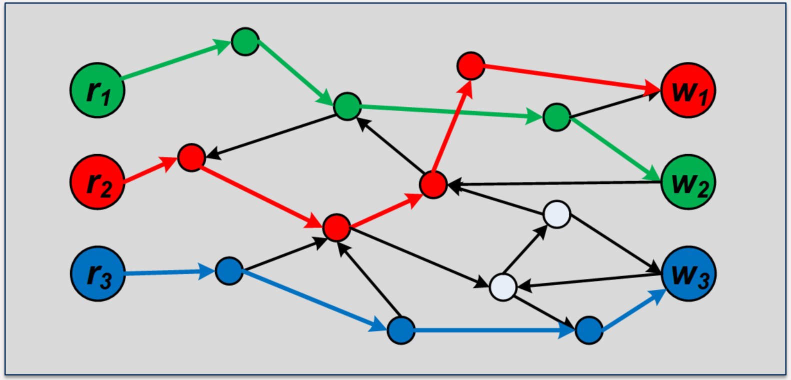
is equal to the maximum number of **vertex disjoint paths** between the sets of inputs and outputs.

A path-based check on the topology of the network model set can decide whether the conditions for identifiability are satisfied generically.

[1] Van der Woude, 1991; [2] Bazanella et al., CDC 2017; [3] Hendrickx et al., 2019.

Generic rank

The **generic rank** of a transfer function can be evaluated by graph-based conditions



Generic rank = number of vertex-disjoint paths

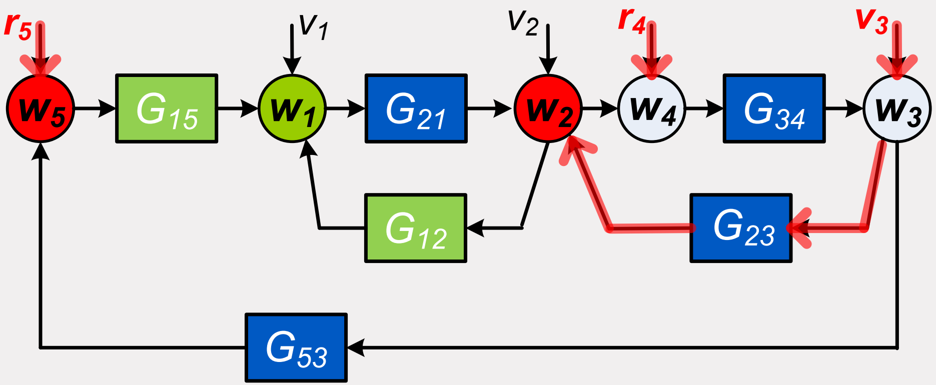
There are graph algorithms for calculating this, based on the topology of the network

No numerical evaluation based on dynamic systems coefficients.

TU/e

Example 5-node network

Verifying the rank condition for w_1:



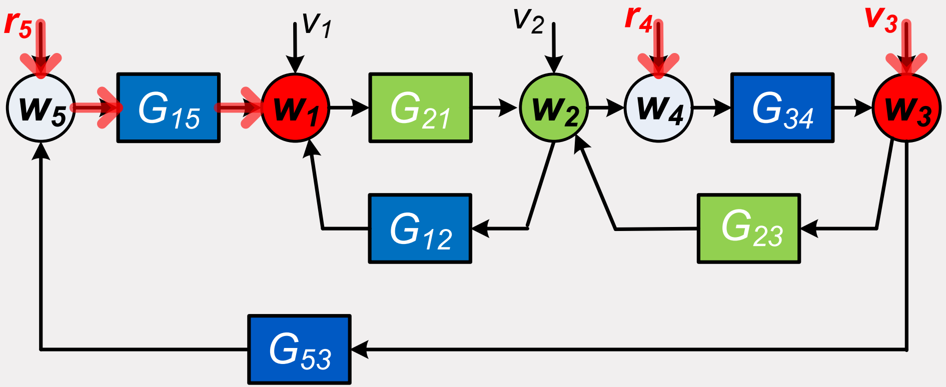
2 vertex-disjoint paths → full row rank 2

Full row rank of

$$\begin{bmatrix} v_3 \\ r_4 \\ r_5 \end{bmatrix} \to \begin{bmatrix} w_2 \\ w_5 \end{bmatrix}$$

Example 5-node network

Verifying the rank condition for w_2:



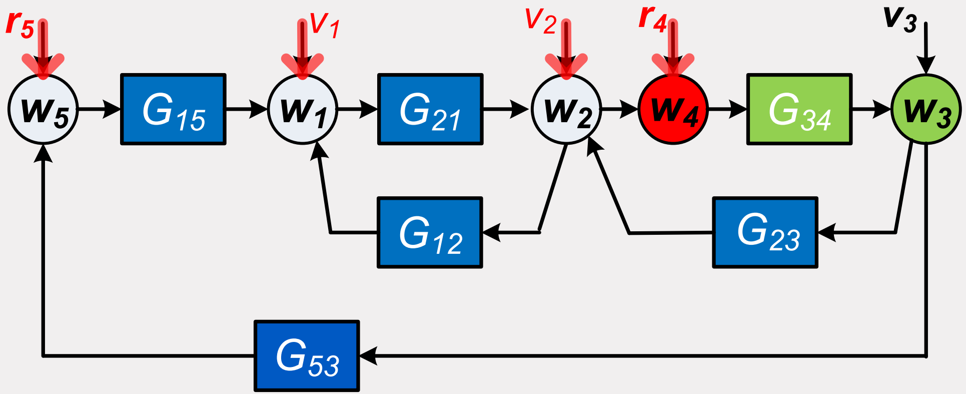
Full row rank of

$$\begin{bmatrix} v_3 \\ r_4 \\ r_5 \end{bmatrix} \rightarrow \begin{bmatrix} w_1 \\ w_3 \end{bmatrix}$$

2 vertex-disjoint paths → full row rank 2 ✓

Example 5-node network

Verifying the rank condition for w_3:



Full row rank of

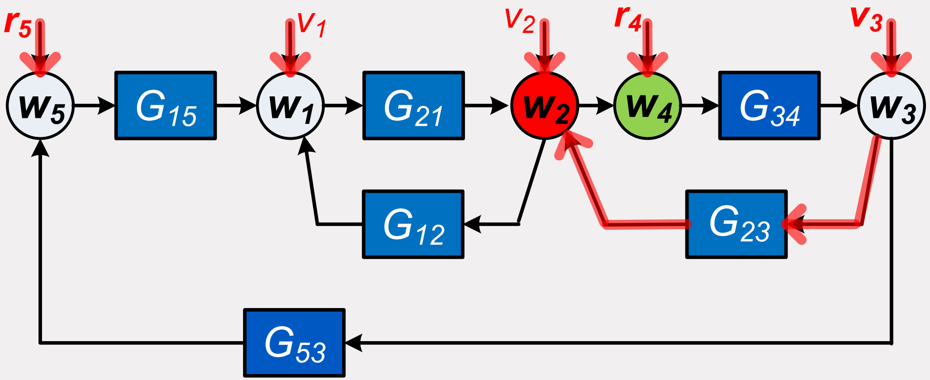
$$\begin{bmatrix} v_1 \\ v_2 \\ r_4 \\ r_5 \end{bmatrix} \rightarrow \begin{bmatrix} w_4 \end{bmatrix}$$

1 vertex-disjoint path → full row rank 1

TU/e

Example 5-node network

Verifying the rank condition for w_4:



Full row rank of

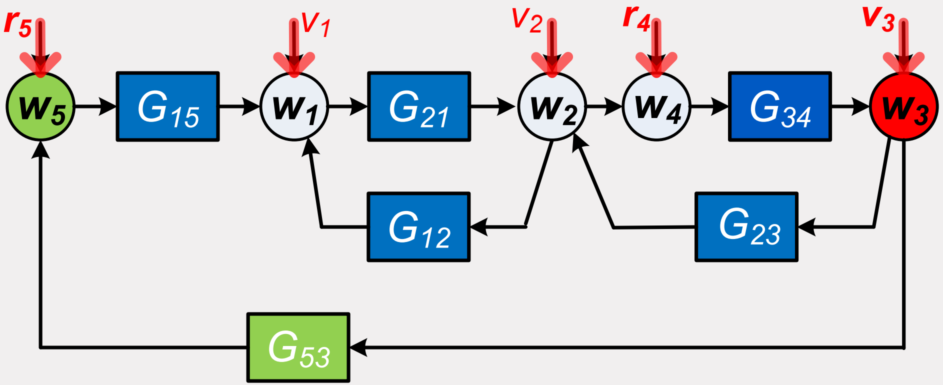
$$\begin{bmatrix} v_1 \\ v_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} \rightarrow \begin{bmatrix} w_2 \end{bmatrix}$$

1 vertex-disjoint paths → full row rank 1 ✓

TU/e

Example 5-node network

Verifying the rank condition for w_5:



Full row rank of

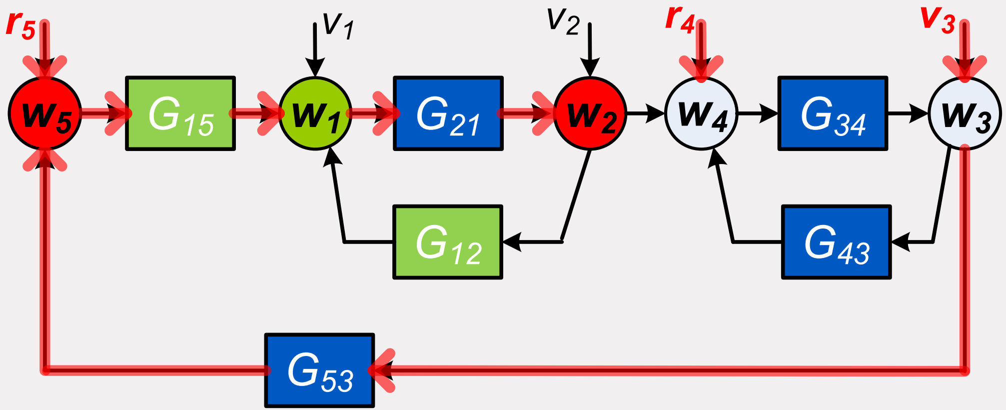
$$\begin{bmatrix} v_1 \\ v_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} \to \begin{bmatrix} w_3 \end{bmatrix}$$

1 vertex-disjoint paths → full row rank 1

Conclusion 5-node example

The structured model set is generically identifiable

If the feedback connection $w_3 \rightarrow w_2$ were to be changed to $w_3 \rightarrow w_4$, then lack of identifiability occurs for the situation $j = 1$



Generic identifiability

Result provides an **analysis tool**, but is less suited for the **synthesis** question:

Given a parametrized network model set:

Where to add external excitation signals to reach generic network identifiability of the full network?

Problem is the "merging" of the results for all $j = 1, \cdots L$

Identifiability concept

We started with three different network identifiability concepts[1]:

(a) Global identifiability at $M(\theta_0)$

(b) Global identifiability

(c) Generic identifiability

In an identification setting, we do not know the system, so concept (a) is less relevant;

With concepts (b) and (c), identifiability is a verifiable property of a model set, rather than an assumption on the underlying system.

[1] Legat and Hendrickx (CDC 2020), introduced the concept of local (generic) identifiability.

Discussion identifiability

Identifiability of a network model property

- Rather than focusing on the full network model, a model **property** can be taken as object for identifiability

$$T(q,\theta_0) = T(q,\theta_1) \implies f(M(\theta_0) = f(M(\theta_1))$$

as e.g. one particular module:

$$f(M(\theta)) = G_{ji}(\theta)$$

This will be addressed separately in single module identifiability

Summary identifiability of full network

Identifiability of network model sets is determined by

- Topology of parametrized modules in model set
- Presence and location of external signals, and
- Presence and correlation structure of disturbances

- Two different concepts:
  global (with algebraic conditions) and generic (with path-based conditions)
- Presented results: all node signals w assumed to be measurable
- Fully applicable to the situation $p < L$ (reduced-rank noise)
- Sufficient conditions for different cases:
  full excitation case and general case (dependent on topology of $G(\theta)$)
- Results not yet suited for synthesis